

MULTIPLE POSITIVE SOLUTIONS TO A FOURTH ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT. We study the existence and multiplicity of positive solutions for a nonlinear fourth-order two-point boundary value problem. The approach is based on critical point theorems in conical shells, Krasnoselskii's compression-expansion theorem, and unilateral Harnack type inequalities.

Keywords: Fourth order differential equation; boundary value problem; positive solution; critical point; fixed point.

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1. INTRODUCTION

The fourth-order boundary value problems appear in the elasticity theory describing stationary states of the deflection of an elastic beam. In last decade a lot of studies are devoted to the existence of positive solutions for such problems, applying Leray-Schauder continuation method, the topological degree theory, the fixed point theorems on cones, the critical point theory or the lower and upper solution method (see, for example, [2, 3, 4, 5, 7, 9, 10, 11, 16, 17]).

In this article, we study the existence and multiplicity of positive solutions for nonlinear fourth-order two-point boundary value problem with cantilever boundary conditions. Consider the fourth-order boundary value problem

$$(1.1) \quad \begin{cases} u^{(4)}(t) - f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where the function $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and $f(t, \mathbb{R}_+) \subset \mathbb{R}_+$ for all $t \in [0, 1]$.

Our approach is based on critical point theorems for functionals in conical shells (see [12, 13]) and Krasnoselskii's compression-expansion theorem. As one can see along the paper, the arguments developed here can be applied to other boundary value problems associated to fourth and sixth order differential equations. Because the estimates are connected with the specific boundary conditions, we concentrate on the model problem (1.1).

The paper is organized as follows. In Section 2 we give formulation of critical point theorems in conical shells and Krasnoselskii's Compression-Expansion Theorem. We present also the variational formulation of the problem. In Section 3, the main existence and multiplicity results Theorems 3.2, 3.3, 3.5 and 3.6 are formulated and proved. Their proofs are based on the mentioned above theorems and inequalities proved in Lemma 3.1 and Lemma 3.4. In order to illustrate the obtained results, two examples are given.

2. PRELIMINARIES

2.1. Critical point theorems in conical shells. In this subsection we introduce the results given in [12] which we are going to apply to the fourth order problem (1.1).

We consider two real Hilbert spaces, X with inner product and norm (\cdot, \cdot) , $|\cdot|$, H with inner product and norm $\langle \cdot, \cdot \rangle$, $\|\cdot\|$, and we assume that $X \subset H$ with continuous injection. We identify H to its dual H' and we obtain $X \subset H \equiv H' \subset X'$. By $\langle \cdot, \cdot \rangle$ we also denote the duality between X and X' , i.e. $\langle x^*, x \rangle = x^*(x)$ for $x^* \in X'$ and $x \in X$. If $x^* \in H$, then $\langle x^*, x \rangle$ is exactly the scalar product in H and $\langle x^*, x \rangle = (x^*, x)$.

We also consider a cone in X , i.e. a convex closed nonempty set K , $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. Let $\phi \in K \setminus \{0\}$ be a fixed element with $|\phi| = 1$. Then, for all numbers R_0, R_1 with $0 < R_0 < \|\phi\| R_1$, there is $\mu > 0$ such that $\|\mu\phi\| > R_0$ and $|\mu\phi| < R_1$. Denote by $K_{R_0 R_1}$ the conical shell

$$K_{R_0 R_1} = \{u \in K : \|u\| \geq R_0, |u| \leq R_1\}.$$

Clearly $\mu\phi$ is an interior point of $K_{R_0 R_1}$, in the sense that $\|\mu\phi\| > R_0$ and $|\mu\phi| < R_1$.

Let L be the continuous linear operator from X to X' , given by

$$(u, v) = \langle L u, v \rangle \quad \text{for all } u, v \in X,$$

and let J from X' into X be the inverse of L . Then

$$(J u, v) = \langle u, v \rangle \quad \text{for } u \in X', v \in X.$$

Let E be a C^1 functional defined on X . We say that E satisfies the *modified Palais-Smale-Schechter condition* (MPSS) in $K_{R_0 R_1}$, if any sequence (u_k) of elements of $K_{R_0 R_1}$ for which $(E(u_k))$ converges and one of the following conditions holds:

- (i) $E'(u_k) \rightarrow 0$;
- (ii) $\|u_k\| = R_0$, $(JE'(u_k), Ju_k) \geq 0$ and $JE'(u_k) - \frac{(JE'(u_k), Ju_k)}{|Ju_k|^2} Ju_k \rightarrow 0$;
- (iii) $|u_k| = R_1$, $(JE'(u_k), u_k) \leq 0$ and $JE'(u_k) - \frac{(JE'(u_k), u_k)}{R_1^2} u_k \rightarrow 0$,

has a convergent subsequence.

We say that E satisfies the *compression boundary condition* in $K_{R_0 R_1}$ if

$$(2.1) \quad JE'(u) - \lambda Ju \neq 0 \quad \text{for } u \in K_{R_0 R_1}, \|u\| = R_0, \lambda > 0;$$

$$(2.2) \quad JE'(u) + \lambda u \neq 0 \quad \text{for } u \in K_{R_0 R_1}, |u| = R_1, \lambda > 0.$$

We say that E has a *mountain pass geometry* in $K_{R_0 R_1}$ if there exist u_0 and u_1 in the same connected component of $K_{R_0 R_1}$, and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in K_{R_0 R_1}, |u| = r\}.$$

In this case we consider the set

$$(2.3) \quad \Gamma = \{\gamma \in C([0, 1]; K_{R_0 R_1}) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and the number

$$(2.4) \quad c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E(\gamma(t)).$$

Finally, we say that E is *bounded from below* in $K_{R_0 R_1}$ if

$$(2.5) \quad m := \inf_{u \in K_{R_0 R_1}} E(u) > -\infty.$$

We assume that the following conditions are satisfied:

$$(2.6) \quad (I - JE')(K) \subset K \quad (I \text{ is the identity map on } X);$$

and there exists a constant $\nu_0 > 0$ such that

$$(2.7) \quad (JE'(u), Ju) \leq \nu_0 \quad \text{for all } u \in K \text{ with } \|u\| = R_0;$$

$$(2.8) \quad (JE'(u), u) \geq -\nu_0 \quad \text{for all } u \in K \text{ with } |u| = R_1.$$

The following theorems of localization of critical points in a conical shell appear as slight particularizations of the main results from [12, 13].

Theorem 2.1. *Assume that E is bounded from below in $K_{R_0 R_1}$ and that there is a $\rho > 0$ with*

$$(2.9) \quad E(u) \geq m + \rho$$

(m given in (2.5)) for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$. In addition assume that E satisfies the (MPSS) condition and the compression boundary condition in $K_{R_0 R_1}$. Then there exists $u \in K_{R_0 R_1}$ such that

$$E'(u) = 0 \quad \text{and} \quad E(u) = m.$$

Theorem 2.2. *Assume that E has the mountain pass geometry in $K_{R_0 R_1}$ and that there is a $\rho > 0$ with*

$$(2.10) \quad |E(u) - c| \geq \rho$$

(c given in (2.4)) for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$. In addition assume that E satisfies the (MPSS) condition and the compression boundary condition in $K_{R_0 R_1}$. Then there exists $u \in K_{R_0 R_1}$ such that

$$E'(u) = 0 \quad \text{and} \quad E(u) = c.$$

Remark 2.1. If the assumptions of both Theorems 2.1, 2.2 are satisfied, since $m < c$, then E has two distinct critical points in $K_{R_0 R_1}$.

2.2. Krasnoselskii's compression-expansion theorem. The problem (1.1) can also be investigated by means of fixed point techniques. In this paper, we are mainly concerned with the variational approach based on critical point theory. However, it deserves to comment about the applicability of fixed point methods and the surplus of information given by the variational approach.

Thus we shall report on the applicability of Krasnoselskii's compression-expansion theorem (see [6, 8]), which guarantees the existence of a fixed point of a compact operator in a conical shell of a Banach space.

Theorem 2.3 (Krasnoselskii). *Let $(X, |\cdot|)$ be a Banach space and $K \subset X$ a cone. Let R_0, R_1 be two numbers with $0 < R_0 < R_1$, $K_{R_0 R_1} = \{u \in K : R_0 \leq |u| \leq R_1\}$, and let $N : K_{R_0 R_1} \rightarrow K$ be a compact operator. Let $<$ be the strict ordering induced in X by the cone K , i.e. $u < v$ if and only if $v - u \in K \setminus \{0\}$. Assume that one of the following conditions is satisfied:*

- (a): *compression:* (i) $N(u) \not\prec u$ for all $u \in K$ with $|u| = R_0$, and (ii) $N(u) \not\prec u$ for all $u \in K$ with $|u| = R_1$;
- (b): *expansion:* (i) $N(u) \succ u$ for all $u \in K$ with $|u| = R_0$, and (ii) $N(u) \prec u$ for all $u \in K$ with $|u| = R_1$.

Then N has at least one fixed point in $K_{R_0 R_1}$.

2.3. Variational formulation of the problem. Next we are going to describe the variational structure of the problem (1.1) (see [15, 16]).

Let X be the Hilbert space

$$X := \{u \in H^2(0, 1) : u(0) = u'(0) = 0\}$$

with inner product and norm

$$(u, v) := \int_0^1 u''(t)v''(t)dt,$$

$$(2.11) \quad |u| := \left(\int_0^1 (u''(t))^2 dt \right)^{\frac{1}{2}}.$$

We associate to the problem (1.1), the functional $E : X \rightarrow \mathbb{R}$ defined by

$$E(u) := \frac{1}{2} |u|^2 - \int_0^1 F(t, u(t)) dt,$$

where

$$F(t, u) = \int_0^u f(t, s) ds.$$

The functional $E : X \rightarrow \mathbb{R}$ is C^1 and for any $u, v \in X$,

$$\langle E'(u), v \rangle = \int_0^1 (u''(t)v''(t) - f(t, u(t))v(t)) dt.$$

Also, $u \in X$ is a critical point of E if and only if u is a classical solution of the problem (1.1) (see [15]).

In this specific case, $H = L^2(0, 1)$ with the usual norm denoted by $\|\cdot\|$, and $L : X \rightarrow X'$ is given by $Lu = u^{(4)}$ (in the distributional sense). The inverse of L is the operator $J : X' \rightarrow X$ which at each $v \in X'$ attaches the unique $u \in X$ with $u^{(4)} = v$ in the sense of distributions.

In particular, if $v \in L^2(0, 1)$, one has the representation

$$(Jv)(t) = \int_0^1 G(t, s)v(s) ds,$$

where $G(t, s)$ is the Green's function related to the fourth order problem

$$\begin{cases} u^{(4)}(t) = v(t), & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

By means of the Mathematica package developed in [1], we have that such function is given by the following expression:

$$(2.12) \quad G(t, s) = \begin{cases} \frac{s^2}{6}(3t - s), & 0 \leq s \leq t \leq 1, \\ \frac{t^2}{6}(3s - t), & 0 \leq t < s \leq 1. \end{cases}$$

Then the problem (1.1) is equivalent to the integral equation

$$(2.13) \quad u(t) = \int_0^1 G(t, s)f(s, u(s)) ds, \quad u \in C[0, 1].$$

We note that $E'(u) = Lu - f(\cdot, u)$ and $JE'(u) = u - N(u)$, where

$$N(u) = Jf(\cdot, u) = \int_0^1 G(\cdot, s) f(s, u(s)) ds.$$

Obviously, (2.13) represents a fixed point equation associated to N . Note also that, since the embedding $X \subset C([0, 1])$ is compact and f is a continuous function, N is a compact operator from X to X .

3. MAIN RESULTS

3.1. Localization in a shell defined by the energetic norm. First we shall deal with the localization of positive solutions u of the problem (1.1) in a shell defined by a single norm, more exactly

$$R_0 \leq |u| \leq R_1,$$

where $|\cdot|$ is the energetic norm given by (2.11). For this, the following unilateral Harnack inequality is crucial.

Lemma 3.1. *If $u \in C^4[0, 1]$ satisfies $u(0) = u'(0) = u''(1) = u'''(1) = 0$ and $u^{(4)}$ is nonnegative and nondecreasing in $[0, 1]$, then u is convex and*

$$(3.1) \quad u(t) \geq M_0(t) |u| \quad \text{for all } t \in [0, 1],$$

where $M_0(t) = \frac{\sqrt{2}(1-t)t^3}{6}$.

Proof. From $u^{(4)} \geq 0$ it follows that u'' is convex. This together with $u''(1) = (u'')'(1) = 0$ gives that u'' is nonnegative and nonincreasing. Next, from $u'' \geq 0$ one has that u is convex, and since $u(0) = u'(0) = 0$, u must be nondecreasing and nonnegative.

On the other hand, since $u^{(4)} \geq 0$ we have that u''' is nondecreasing and since $u'''(1) = 0$, $u''' \leq 0$. Then u' is concave; it is also nondecreasing due to $u'' \geq 0$, and since $u'(0) = 0$, we have $u' \geq 0$. Now from $u'' \geq 0$, $u' \geq 0$ and $u(0) = 0$, we see that u is nonnegative, nondecreasing and convex.

Finally note that from $u^{(4)}$ nondecreasing, we have that u''' is convex, and since $u'''(1) = 0$, the graph of u''' is under the line connecting the points $(0, u'''(0))$ and $(1, 0)$, i.e.

$$(3.2) \quad u'''(t) \leq (1-t) u'''(0), \quad t \in [0, 1].$$

Due to the fact that the function u'' is nonincreasing and the function u''' is nondecreasing we have:

$$\begin{aligned} u(t) &= \int_0^t \int_0^s u''(\tau) d\tau ds \geq \int_0^t \int_0^s u''(s) d\tau ds = \int_0^t s u''(s) ds \\ &= \frac{t^2}{2} u''(t) - \int_0^t \frac{s^2}{2} u'''(s) ds \geq - \int_0^t \frac{s^2}{2} u'''(s) ds \\ &\geq - \int_0^t \frac{s^2}{2} u'''(t) ds = - \frac{t^3}{6} u'''(t). \end{aligned}$$

This inequality combined with (3.2) gives

$$(3.3) \quad u(t) \geq - \frac{(1-t)t^3}{6} u'''(0).$$

Next we deal with the energetic norm wishing to connect it to $u'''(0)$. One has

$$(3.4) \quad \begin{aligned} |u|^2 &= \int_0^1 u''(t)^2 dt = u''u'|_0^1 - \int_0^1 u'''(t) u'(t) dt \\ &= - \int_0^1 u'''(t) u'(t) dt \leq -u'''(0) u'(1). \end{aligned}$$

Also

$$(3.5) \quad \begin{aligned} u'(1) &= \int_0^1 u''(t) dt = - \int_0^1 \int_t^1 u'''(s) ds dt \leq - \int_0^1 \int_t^1 u'''(t) ds dt \\ &= - \int_0^1 (1-t) u'''(t) dt \leq - \int_0^1 (1-t) u'''(0) dt = -\frac{1}{2} u'''(0). \end{aligned}$$

From (3.4) and (3.5) we deduce $|u|^2 \leq \frac{1}{2} u'''(0)^2$, or

$$-u'''(0) \geq \sqrt{2} |u|.$$

This inequality and (3.3) prove (3.1). \square

Consider the cone

$$K := \{u \in X : u \text{ convex and } u(t) \geq M_0(t) |u| \text{ on } [0, 1]\}.$$

We note that, since any convex function with $u(0) = u'(0) = 0$ is nondecreasing, all the elements of K are nondecreasing functions.

Also $K \neq \{0\}$. Indeed, if we consider the eigenvalue problem

$$\begin{cases} \phi^{(4)}(t) = \lambda \phi(t), & 0 < t < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0, \end{cases}$$

then its first eigenvalue $\lambda_1 = \beta^4$, where $\beta = \frac{\pi}{2} + 0.3042$ is the smallest positive solution of the equation

$$\cos \lambda \cosh \lambda + 1 = 0,$$

while the function

$$(3.6) \quad \phi_1(t) = \sin \beta t - \sinh \beta t + \frac{\sinh \beta + \sin \beta}{\cosh \beta + \cos \beta} (\cosh \beta t - \cos \beta t)$$

is a positive eigenfunction (see [14]) corresponding to λ_1 . In addition, one can check that $\phi_1'' \geq 0$, that is ϕ_1 is convex. Since $\phi_1(0) = \phi_1'(0) = 0$, we must have $\phi_1 \geq 0$ and $\phi_1' \geq 0$ on $[0, 1]$. As consequence, $\phi_1^{(5)} = \lambda \phi_1' \geq 0$ on $[0, 1]$. Then, according to Lemma 3.1, $\phi_1 \in K$.

Denote

$$M_1(t) := \frac{2}{3} t^{\frac{3}{2}}, \quad t \in [0, 1].$$

Our assumptions on f are as follows:

(h1): f is nondecreasing on $[0, 1] \times \mathbb{R}_+$ in each of its variables;

(h2): there exist R_0, R_1 with $0 < R_0 < R_1$ such that

(a): $\int_0^1 M_0(t) f(t, M_0(t) R_0) dt \geq R_0$,

(b): $\int_0^1 M_1(t) f(t, M_1(t) R_1) dt \leq R_1$.

(h3): there exist $u_0, u_1 \in K_{R_0 R_1} = \{u \in K : R_0 \leq |u| \leq R_1\}$ and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max \{E(u_0), E(u_1)\} < \inf \{E(u) : u \in K, |u| = r\}.$$

Theorem 3.2. Assume that (h1), (h2) are satisfied. Let Γ , m and c be defined as in (2.3), (2.4) and (2.5) respectively. Then the fourth-order problem (1.1) has at least one positive solution u_m in $K_{R_0 R_1}$ such that

$$E(u_m) = m.$$

If in addition (h3) holds, then a second positive solution u_c exists in $K_{R_0 R_1}$ with

$$E(u_c) = c.$$

Proof. First let us note that the (MPSS) condition holds in $K_{R_0 R_1}$ due to the compactness of the operator $N = I - JE'$. Also the boundedness of $(JE'(u), Ju)$ and $(JE'(u), u)$ on the boundaries of $K_{R_0 R_1}$, i.e. (2.7) and (2.8) is guaranteed since JE' maps bounded sets into bounded sets.

To check (2.6), let u be any element of K . Hence u is nonnegative and non-decreasing on $[0, 1]$. Then, from (h1) we also have that $f(t, u(t))$ is nonnegative and nondecreasing in $[0, 1]$. Now, Lemma 3.1 implies that $Jf(\cdot, u(\cdot)) \in K$. But $Jf(\cdot, u(\cdot)) = (I - JE')(u)$. Thus (2.6) holds.

Next, let us note that for any $u \in K_{R_0 R_1}$, we have

$$\begin{aligned} (3.7) \quad u(t) &= \int_0^t \int_0^s u''(\tau) d\tau ds \leq \int_0^t \sqrt{s} \left(\int_0^s u''(\tau)^2 d\tau \right)^{1/2} ds \\ &\leq |u| \int_0^t \sqrt{s} ds = \frac{2}{3} t^{3/2} |u| = M_1(t) |u|. \end{aligned}$$

Then

$$E(u) = \frac{1}{2} |u|^2 - \int_0^1 F(t, u(t)) dt \geq \frac{1}{2} R_0^2 - F\left(1, \frac{2}{3} R_1\right).$$

Hence E is bounded from below on $K_{R_0 R_1}$.

Furthermore, we check the boundary conditions (2.1). Assume that $JE'(u) - \lambda Ju = 0$ for some $u \in K$ with $|u| = R_0$ and $\lambda > 0$. Then u solves the problem

$$\begin{cases} u^{(4)}(t) - f(t, u(t)) - \lambda u(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

and

$$\begin{aligned} R_0^2 &= |u|^2 = \int_0^1 [f(t, u(t)) + \lambda u(t)] u(t) dt \\ &\geq \int_0^1 [f(t, M_0(t) R_0) + \lambda M_0(t) R_0] M_0(t) R_0 dt \\ &> R_0 \int_0^1 f(t, M_0(t) R_0) M_0(t) dt, \end{aligned}$$

which contradicts the assumption (h2) (a). Hence $JE'(u) - \lambda Ju \neq 0$ for all $u \in K$ with $|u| = R_0$ and $\lambda > 0$.

Assume now that $JE'(u) + \lambda u = 0$ for some $u \in K$ with $|u| = R_1$ and $\lambda > 0$. Then u solves the problem

$$\begin{cases} (1 + \lambda) u^{(4)}(t) - f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

Then

$$R_1^2 = |u|^2 = \frac{1}{1 + \lambda} \int_0^1 f(t, u(t)) u(t) dt.$$

Using (3.7) we deduce

$$R_1^2 < \int_0^1 M_1(t) R_1 f(t, M_1(t) R_1) dt,$$

which contradicts (h2) (b). Hence $JE'(u) + \lambda u \neq 0$ for all $u \in K$ with $|u| = R_1$ and $\lambda > 0$.

The conclusions follow from Theorem 2.1 and Theorem 2.2. \square

For the autonomous case $f(t, u) = f(u)$, where f is nonnegative and nondecreasing on \mathbb{R}_+ , we may replace the conditions of (h2) by a couple of simpler inequalities.

Example 3.1. We give an example of a function $f(u)$ which satisfies the conditions (h_1) and (h_2) of Theorem 3.2. Note that

$$0 \leq \frac{\sqrt{2}}{6}(1-t)t^3 < 0.03 \quad \text{if } 0 \leq t \leq 1,$$

$$\int_0^1 \left(\frac{\sqrt{2}}{6}(1-t)t^3 \right)^2 dt = \frac{1}{4536},$$

and $4600 \times \frac{3}{100} = 138$. Define

$$f(u) = \begin{cases} 0, & u \leq 0, \\ 4600u, & 0 \leq u \leq 0.03, \\ 138, & u \geq 0.03. \end{cases}$$

Taking $R_0 = 1$ and $R_1 = 37$, by

$$\int_0^1 138 \frac{2}{3} t^{\frac{3}{2}} dt = \frac{184}{5} = 36.8,$$

we obtain that the conditions (h_1) and (h_2) are satisfied.

Further, we have:

Theorem 3.3. Assume that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous nondecreasing and that for some numbers $a \in (0, 1)$, R_0 and R_1 with $0 < R_0 < R_1$, one has

$$(3.8) \quad \frac{f(M_0(a)R_0)}{M_0(a)R_0} \geq \frac{1}{(1-a)M_0(a)^2}, \quad \frac{f\left(\frac{2}{3}R_1\right)}{R_1} \leq \frac{15}{4}.$$

Then (1.1) has at least one positive solution u_m in $K_{R_0R_1}$ with $E(u_m) = m$. If in addition (h3) holds, then a second positive solution u_c exists in $K_{R_0R_1}$ with $E(u_c) = c$.

Proof. Since $M_1(t) \leq 2/3$ for every $t \in [0, 1]$, we have

$$\int_0^1 M_1(t) f(M_1(t)R_1) dt \leq f\left(\frac{2}{3}R_1\right) \int_0^1 \frac{2}{3} t^{\frac{3}{2}} dt = \frac{4}{15} f\left(\frac{2}{3}R_1\right).$$

Then the inequality

$$\frac{4}{15} f\left(\frac{2}{3}R_1\right) \leq R_1,$$

or equivalently the second inequality in (3.8) is a sufficient condition for (h2)(b) to hold. As concerns the first inequality in (3.8), let us remark that if $JE'(u) - \lambda Ju = 0$ for some $u \in K$ with $|u| = R_0$ and $\lambda > 0$, then

$$(3.9) \quad \begin{aligned} R_0^2 &= |u|^2 = \int_0^1 [f(u(t)) + \lambda u(t)] u(t) dt \\ &\geq \int_a^1 [f(u(t)) + \lambda u(t)] u(t) dt. \end{aligned}$$

The function u being nondecreasing, one has $u(t) \geq u(a)$ for all $t \in [a, 1]$. Also, from (3.1), $u(a) \geq M_0(a)|u|$. Then from (3.9),

$$\begin{aligned} R_0^2 &\geq (1-a)[f(M_0(a)R_0) + \lambda M_0(a)R_0] M_0(a)R_0 \\ &> (1-a)f(M_0(a)R_0) M_0(a)R_0. \end{aligned}$$

Hence

$$R_0 > (1-a)M_0(a)f(M_0(a)R_0),$$

i.e. the opposite of the first inequality in (3.8). \square

Clearly the inequalities (3.8) express the oscillation of the function $f(t)/t$ up and down the values $1/(1-a)M_0(a)^2$ and $45/8$.

Remark 3.1 (Existence asymptotic conditions). The existence of two numbers R_0, R_1 satisfying (3.8) is guaranteed by the asymptotic conditions

$$\limsup_{\tau \rightarrow 0} \frac{f(\tau)}{\tau} > \frac{1}{(1-a)M_0(a)^2} \quad \text{and} \quad \liminf_{\tau \rightarrow \infty} \frac{f(\tau)}{\tau} < \frac{45}{8}.$$

Remark 3.2 (Multiplicity). Theorems 3.2 and 3.3 can be used to obtain multiple positive solutions. Indeed, if their assumptions are fulfilled for two pairs (R_0, R_1) , (\bar{R}_0, \bar{R}_1) , then we obtain four solutions, provided that the sets $K_{R_0 R_1}$ and $K_{\bar{R}_0 \bar{R}_1}$ are disjoint. This happens if $0 < R_0 < R_1 < \bar{R}_0 < \bar{R}_1$. We can even obtain sequences of positive solutions; for instance, in connection with Theorem 3.3, if

$$\limsup_{\tau \rightarrow 0} \frac{f(\tau)}{\tau} > \frac{1}{(1-a)M_0(a)^2} \quad \text{and} \quad \liminf_{\tau \rightarrow 0} \frac{f(\tau)}{\tau} < \frac{45}{8},$$

then there exists a sequence (u_k) of positive solutions with $u_k \rightarrow 0$ as $k \rightarrow \infty$. Also, if

$$\limsup_{\tau \rightarrow \infty} \frac{f(\tau)}{\tau} > \frac{1}{(1-a)M_0(a)^2} \quad \text{and} \quad \liminf_{\tau \rightarrow \infty} \frac{f(\tau)}{\tau} < \frac{45}{8},$$

then there exists a sequence (u_k) of positive solutions with $|u_k| \rightarrow \infty$ as $k \rightarrow \infty$.

Remark 3.3 (Fixed point approach). Under the assumptions of Theorem 3.2, the existence of a solution in $K_{R_0 R_1}$ can also be obtained via Krasnoselskii's theorem. Indeed, the problem (1.1) is equivalent to the fixed point problem (2.13) in X for the compact operator $N : K_{R_0 R_1} \rightarrow K$, $N(u) = Jf(\cdot, u(\cdot))$.

Let us check the condition (a)(i). Assume the contrary, i.e. $Nu < u$ for some $u \in K$ with $|u| = R_0$. Then $Nu = u - v$ for some $v \in K \setminus \{0\}$. This means that $(u - v)^{(4)} = f(t, u)$ in the sense of distributions. Now multiply by u and integrate to obtain

$$|u|^2 - \int_0^1 u''(t)v''(t)dt = \int_0^1 f(t, u(t))u(t)dt.$$

Since $v, u - v \in K$, one has $v'' \geq 0$ and $u'' - v'' \geq 0$ in $[0, 1]$. Hence

$$\int_0^1 u''(t) v''(t) dt \geq \int_0^1 v''(t)^2 dt = |v|^2 > 0.$$

Then

$$R_0^2 = |u|^2 > \int_0^1 f(t, u(t)) u(t) dt.$$

Next we use (3.1) to derive a contradiction to (h2) (a).

The condition (a)(ii) can be proved similarly.

Notice that under assumptions (h2)(a) and (b), a solution exists in $K_{R_1 R_0}$ in case that $R_1 < R_0$. However this is not guaranteed by the variational approach.

We may conclude that, compared to the fixed point approach, the variational method gives an additional information about the solution, namely of being a minimum for the energy functional. Moreover, a second solution of mountain pass type can be guaranteed by the variational approach.

The above approach was essentially based on the monotonicity assumption on f , which was required by the Harnack type inequality (3.1). Thus a natural question is if such an inequality can be established for functions u satisfying the boundary conditions and $u^{(4)} \geq 0$, without the assumption that $u^{(4)}$ is nondecreasing. In the absence of the answer to this question, an alternative approach is possible in a shell defined by two norms as shown in the next section.

3.2. Localization in a shell defined by two norms. In the previous section, a unilateral Harnack inequality was established for functions u satisfying the two point boundary conditions and with $u^{(4)}$ nonnegative and nondecreasing in $[0, 1]$, in terms of the energetic norm. If we renounce to the monotonicity of $u^{(4)}$, then we have the following result in terms of the max norm.

Lemma 3.4. *If $u \in C^4[0, 1]$ satisfies $u(0) = u'(0) = u''(1) = u'''(1) = 0$ and $u^{(4)} \geq 0$ in $[0, 1]$, then*

$$(3.10) \quad u(t) \geq M(t) \|u\|_\infty \quad \text{for all } t \in [0, 1],$$

$$\text{where } M(t) := \frac{(3-t)t^2}{3}.$$

Proof. Fix $t \in (0, 1)$. Considering the expression of the Green's function given in (2.12), we have

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{G(t, s)}{s^2} &= \lim_{s \rightarrow 0^+} \frac{3t - s}{6} = \frac{t}{2} > 0, \\ \lim_{s \rightarrow 1^-} \frac{G(t, s)}{s^2} &= \lim_{s \rightarrow 1^-} \frac{t^2}{6} \left(\frac{3}{s} - \frac{t}{s^2} \right) = \frac{t^2}{6} (3 - t) > 0. \end{aligned}$$

Then, we can affirm that for every $t \in (0, 1)$ fixed, the function $\frac{G(t, s)}{s^2}$ is continuous and positive on $[0, 1]$ and it has strictly positive maximum and minimum on $[0, 1]$.

Let

$$(3.11) \quad H(t, s) = \frac{G(t, s)}{s^2} = \begin{cases} H_1(t, s) = \frac{3t - s}{6}, & 0 \leq s \leq t \leq 1, \\ H_2(t, s) = \frac{t^2}{6} \left(\frac{3}{s} - \frac{t}{s^2} \right), & 0 \leq t < s \leq 1, \end{cases}$$

We have

$$\frac{\partial}{\partial s} H_1(t, s) = -\frac{1}{6} < 0.$$

Then $H_1(t, s)$ is a decreasing function for $s \in [0, t]$; hence we have that

$$H_1(t, s) \leq H_1(t, 0) = \frac{t}{2} \quad \text{for } 0 \leq s \leq t \leq 1.$$

Also, we obtain

$$\frac{\partial}{\partial s} H_2(t, s) = \frac{t^2}{6} \left(-\frac{3}{s^2} + \frac{2t}{s^3} \right)$$

and

$$\frac{\partial^2}{\partial s^2} H_2(t, s) = \frac{t^2}{s^4} (s - t) \geq 0, \quad 0 \leq t < s \leq 1,$$

then

$$\frac{\partial}{\partial s} H_2(t, s) \leq \frac{\partial}{\partial s} H_2(t, s)|_{s=1} = \frac{t^2}{6} (2t - 3) < 0,$$

hence

$$H_2(t, s) \geq H_2(t, 1) = \frac{t^2}{6} (3 - t) \quad \text{for } 0 \leq t < s \leq 1.$$

Combining the previous results we obtain the following estimations for the Green function

$$(3.12) \quad \frac{t^2}{6} (3 - t) s^2 \leq G(t, s) \leq \frac{s^2}{2} \quad \text{for all } (t, s) \in [0, 1] \times [0, 1].$$

Let $u \in C^4[0, 1]$ satisfy $u(0) = u'(0) = u''(1) = u'''(1) = 0$ and $u^{(4)} \geq 0$ in $[0, 1]$. Then

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) u^{(4)}(s) ds \geq \int_0^1 \frac{(3-t)t^2 s^2}{6} u^{(4)}(s) ds \\ &= \frac{(3-t)t^2}{6} \int_0^1 s^2 u^{(4)}(s) ds \geq \frac{(3-t)t^2}{3} \int_0^1 \left\{ \max_{t \in [0, 1]} G(t, s) \right\} u^{(4)}(s) ds \\ &\geq \frac{(3-t)t^2}{3} \max_{t \in [0, 1]} \left\{ \int_0^1 G(t, s) u^{(4)}(s) ds \right\} = \frac{(3-t)t^2}{3} \|u\|_\infty. \end{aligned}$$

□

Notice that, since $\|u\| \leq \|u\|_\infty$, the inequality (3.10) also gives

$$u(t) \geq M(t) \|u\| \quad \text{for all } t \in [0, 1].$$

Using Lemma 3.4, the existence of a positive solution can be immediately obtained via Krasnoselskii's theorem.

Theorem 3.5. Assume that there exist positive numbers α, β , $\alpha \neq \beta$ such that

$$(3.13) \quad \alpha \leq (J \underline{f}_\alpha)(1) \quad \text{and} \quad \beta \geq (J \overline{f}_\beta)(1),$$

where

$$\begin{aligned} \underline{f}_\alpha(t) &= \min \{ f(t, u) : M(t) \alpha \leq u \leq \alpha \}, \\ \overline{f}_\beta(t) &= \max \{ f(t, u) : M(t) \beta \leq u \leq \beta \}. \end{aligned}$$

Then the problem (1.1) has at least one positive solution u such that

$$R_0 \leq \|u\|_\infty \leq R_1,$$

where $R_0 = \min \{\alpha, \beta\}$, $R_1 = \max \{\alpha, \beta\}$.

Proof. The problem (1.1) is equivalent to the fixed point problem $N(u) = u$ in $C[0, 1]$, where $N(u) = Jf(\cdot, u(\cdot))$.

In the space $C[0, 1]$ we consider the cone

$$K = \{u \in C[0, 1] : u(0) = 0, u(t) \geq M(t) \|u\|_\infty \text{ for all } t \in [0, 1]\}.$$

According to inequalities (3.12), it is not difficult to verify that $N(K) \subset K$. Also N is a compact operator.

Now we show that the required boundary conditions from Krasnoselskii's theorem are satisfied. Assume by contradiction that $Nu < u$ for some $u \in K$ with $\|u\|_\infty = \alpha$. Then $Nu = u - v$ for some $v \in K \setminus \{0\}$. Hence

$$(3.14) \quad u(t) - v(t) = Jf(t, u(t)).$$

We have

$$M(t)\alpha \leq u(t) \leq \alpha \quad \text{for all } t \in [0, 1].$$

Hence

$$f(t, u(t)) \geq \underline{f}_\alpha(t).$$

Since J is a positive linear operator, it preserves ordering, so $Jf(t, u(t)) \geq (J\underline{f}_\alpha)(t)$. Returning to (3.14), we deduce that

$$(3.15) \quad u(t) - v(t) \geq (J\underline{f}_\alpha)(t).$$

Since $v(t) \geq M(t) \|v\|_\infty > 0$, for $t > 0$, (3.15) yields

$$\alpha = \|u\|_\infty \geq u(1) > u(1) - v(1) \geq (J\underline{f}_\alpha)(1),$$

a contradiction to our assumption.

Next assume that $Nu > u$ for some $u \in K$ with $\|u\|_\infty = \beta$. Then $Nu = u + v$ for some $v \in K \setminus \{0\}$ and, since $G(t, s) \leq G(1, s)$ for all $t, s \in [0, 1]$, we have

$$(3.16) \quad u(t) + v(t) = Jf(t, u(t)) \leq (J\overline{f}_\beta)(t) \leq (J\overline{f}_\beta)(1).$$

Let t_0 be such that $u(t_0) = \|u\|_\infty = \beta > 0$. Since $u(0) = 0$, one has $t_0 > 0$ and so $v(t_0) \geq M(t_0) \|v\|_\infty > 0$. Then, for $t = t_0$, (3.16) gives

$$\beta < (J\overline{f}_\beta)(1),$$

which contradicts our assumption. Thus Theorem 2.3 applies.

We note that if $\alpha < \beta$, then (3.13) represents the compression condition, while if $\alpha > \beta$, then (3.13) expresses the expansion condition. \square

Next we are interested into two positive solutions for (1.1). We shall succeed this by the variational approach based on Theorems 2.1 and 2.2 applied in the Hilbert space $X = \{u \in H^2(0, 1) : u(0) = u'(0) = 0\}$ and to the norms $|\cdot|$ (given by (2.11)) and $\|\cdot\| = \|\cdot\|_{L^2(0,1)}$.

Let us consider the cone

$$K = \{u \in X : u \text{ convex and } u(t) \geq M(t) \|u\| \text{ for all } t \in [0, 1]\}$$

and the numbers R_0, R_1 such that $0 < R_0 < \|\phi\| R_1$, where $\phi = \phi_1/|\phi_1|$ and ϕ_1 is the eigenfunction given by (3.6). Also, let

$$K_{R_0 R_1} = \{u \in K : \|u\| \geq R_0, |u| \leq R_1\}.$$

Denote

$$\begin{aligned}\underline{g}(t) &= \min \{f(t, u) : M(t) R_0 \leq u \leq c_\infty R_1\}, \\ \overline{g}(t) &= \max \{f(t, u) : M(t) R_0 \leq u \leq c_\infty R_1\},\end{aligned}$$

where $c_\infty > 0$ is such that $\|v\|_\infty \leq c_\infty |v|$ for all $v \in K$. For example we may take $c_\infty = 2/3$, since for any $v \in K$, Hölder's inequality gives

$$(3.17) \quad v(t) = \int_0^t \int_0^s v''(\tau) d\tau ds \leq |v| \int_0^t \sqrt{s} \leq \frac{2}{3} |v|.$$

Our assumptions are as follows:

(H1): There exist R_0, R_1 with $0 < R_0 < \|\phi\| R_1$ such that

(a): $R_0 \leq \|J\underline{g}\|$,

(b): $R_1 \geq c_\infty \|\overline{g}\|_{L^1(0,1)}$.

(H2): The functional E has the mountain pass geometry in $K_{R_0 R_1}$ and there exists $\rho > 0$ such that

$$(3.18) \quad E(u) \geq c + \rho$$

for all $u \in K_{R_0 R_1}$ which simultaneously satisfy $\|u\| = R_0$ and $|u| = R_1$.

Theorem 3.6. *Under assumptions (H1), (H2), the problem (1.1) has at least two positive solutions $u_m, u_c \in K_{R_0 R_1}$ with $E(u_m) = m$ and $E(u_c) = c$, with m and c defined on (2.5) and (2.4) respectively.*

Proof. For $u \in K_{R_0 R_1}$, one has

$$M(t) R_0 \leq M(t) \|u\| \leq u(t) \leq \|u\|_\infty \leq c_\infty |u| \leq c_\infty R_1.$$

It follows that

$$F(t, u(t)) \leq \omega := \max \{F(t, u) : 0 \leq t \leq 1, M(t) R_0 \leq u \leq c_\infty R_1\},$$

whence, for all $u \in K_{R_0 R_1}$, it is fulfilled that

$$E(u) = \frac{1}{2} |u|^2 - \int_0^1 F(t, u(t)) dt \geq -\omega,$$

and so, $m > -\infty$.

Next, from $c > m$ we see that (3.18) guarantees both (2.9) and (2.10). It remains to check the compression boundary condition given by (2.1), (2.2). Assume first that (2.1) does not hold. Then $JE'(u) - \lambda Ju = 0$ for some $u \in K_{R_0 R_1}$, $\|u\| = R_0$ and $\lambda > 0$.

Then, for $t > 0$,

$$u(t) = J(f(t, u(t)) + \lambda u(t)) > Jf(t, u(t)) \geq (J\underline{g})(t) \geq 0.$$

Taking the L^2 -norm, we deduce

$$R_0 = \|u\| > \|J\underline{g}\|,$$

which contradicts (H1)(a).

Next assume that $JE'(u) + \lambda u = 0$ for some $u \in K_{R_0 R_1}$, $|u| = R_1$ and $\lambda > 0$. Then

$$(1 + \lambda) u^{(4)} = f(t, u(t)),$$

whence, arguing as in the proof of Theorem 3.2, we deduce that

$$(1 + \lambda) R_1^2 = \int_0^1 u(t) f(t, u(t)) dt.$$

Consequently

$$R_1 < c_\infty \|\bar{g}\|_{L^1(0,1)},$$

which contradicts (H1)(b). \square

Remark 3.4. In the autonomous case, if $f = f(u)$, and f is nondecreasing on \mathbb{R}_+ , a sufficient condition for (H1)(a) to hold is

$$R_0 \leq f(M(a) R_0) \|J\chi_{[a,1]}\|,$$

where a is some number from $(0, 1)$ and $\chi_{[a,1]}$ is the characteristic function of the interval $[a, 1]$. Also in this case, (H1)(b) reduces to

$$R_1 \geq c_\infty f(c_\infty R_1).$$

3.3. An example. We are going to see an example inspired by that in [12], to which we can apply Theorem 3.3.

Example 3.2. Let $0 \leq p \leq \frac{1}{2}$ and

$$(3.19) \quad f(t, u) = f(u) = \begin{cases} pu^p, & 0 \leq u \leq 1, \\ pu^2, & 1 \leq u \leq b, \\ p((u-b)^p + b^2), & u \geq b, \end{cases}$$

where $b > 2$ is chosen below.

This function is positive and nondecreasing in \mathbb{R}_+ . For $1 \leq u \leq 2$, we have

$$F(u) = \frac{p}{p+1} + \frac{p}{3}(u^3 - 1) \leq \frac{p}{p+1} + \frac{7p}{3} = \frac{p(10+7p)}{3(p+1)},$$

and since F is nondecreasing,

$$F(u) \leq \frac{p(10+7p)}{3(p+1)} \quad \text{for } 0 \leq u \leq 2.$$

Choose $r = 2$. Then, for $u \in K$ and $|u| = 2$ according to (3.17), $|u|_\infty \leq \frac{2}{3}|u| < 2$ and so, recalling $0 \leq p \leq \frac{1}{2}$,

$$E(u) = \frac{|u|^2}{2} - \int_0^1 F(u(t)) dt \geq 2 - \frac{p(10+7p)}{3(p+1)} \geq \frac{1}{2}.$$

We take for u_0 the normalized function

$$u_0 = \frac{\phi_1}{|\phi_1|}, \quad |u_0| = 1 < r = 2,$$

for which

$$E(u_0) = \frac{|u_0|^2}{2} - \int_0^1 F(u_0(t)) dt = \frac{1}{2} - \int_0^1 F(u_0(t)) dt < \frac{1}{2}.$$

Next we choose $u_1 = b u_0 / \|u_0\|_\infty$. Then $|u_1| = b / \|u_0\|_\infty > 2$ if we choose $b > 2 \|u_0\|_\infty$. Also

$$E(u_1) \leq \frac{b^2}{2 \|u_0\|_\infty^2} - \int_{(u_1 > 1)} F(u_1(t)) dt.$$

Since $\|u_1\|_\infty = b > 2$ and $u_1(0) = 0$ the level set $(u_1 > 1)$ is a proper subset of $[0, 1]$. Also $u_1(t) \leq b$ for all t . Hence on the level set $(u_1 > 1)$ we have

$$F(u_1) = \frac{p}{p+1} + \frac{p}{3}(u_1^3 - 1) > \frac{p}{3}u_1^3.$$

Then

$$E(u_1) < \frac{b^2}{2\|u_0\|_\infty^2} - \frac{pb^3}{3\|u_0\|_\infty^3} \int_{(u_1 > 1)} u_0(t)^3 dt.$$

Taking into account that the level set $(u_1 > 1)$ enlarges as b increases, we can see that the right side of the last inequality tends to $-\infty$ as $b \rightarrow +\infty$. Thus we may choose b large enough to have

$$E(u_1) < \frac{1}{2}.$$

Hence the assumption (h3) of Theorem 3.3 is satisfied. Also, by

$$\lim_{s \rightarrow 0} \frac{f(s)}{s} = +\infty, \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{f(s)}{s} = 0,$$

we may find R_0 (small enough) and R_1 (large enough), such that u_0 and u_1 belong to $K_{R_0 R_1}$ and the conditions (3.8) hold. Therefore, according to Theorem 3.3, the problem (1.1) with f given by (3.19) and b sufficiently large has two positive solutions.

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